

ANALYSIS ON SOME INFINITE MODULES, INNER PROJECTION, AND APPLICATIONS

KANGJIN HAN AND SIJONG KWAK

ABSTRACT. We are interested in the algebraic and geometric structures of inner projections, the partial elimination ideal theory initiated by M. Green and geometric applications. By developing the elimination mapping cone theorem for infinitely generated graded modules and using the induced multiplicative maps, we get interesting relations between syzygies of the original variety and those of inner projection images. As results, first of all, we show that the inner projection from any smooth point of X satisfies at least property $\mathbf{N}_{2,p-1}$ for a projective reduced connected scheme X of codimension e satisfying property $\mathbf{N}_{2,p}$, $p \geq 1$. Further, we obtain the main theorem on ‘embedded linear syzygies’ which is the natural projection-analogue of ‘restricting linear syzygies’ in the linear section case ([10]). We also obtain that the arithmetic depths of inner projections are equal to that of X which is cut out by quadrics. This gives useful information on the rigidity of Betti diagrams.

These uniform behaviors look unusual in a sense that property $\mathbf{N}_{2,p}$ and the arithmetic depths of outer projections depend heavily on moving the center of projection in an ambient space ([6],[22],[24]). Moreover, these properties have many interesting corollaries such as ‘rigidity theorem’ on property $\mathbf{N}_{2,p}$, $e - 1 \leq p \leq e$ as classifications and the sharp lower bound $e \cdot p - \frac{p(p-1)}{2}$ for the number of quadrics vanishing on X with property $\mathbf{N}_{2,p}$, $p \geq 1$. Examples and further questions are suggested.

Keywords: linear syzygies, the mapping cone sequence, partial elimination ideals, inner projection, arithmetic depth, Castelnuovo-Mumford regularity.

CONTENTS

1. Introduction	1
2. Graded mapping cone construction and Partial elimination ideals	4
3. Embedded linear syzygies, depth and applications	9
4. Some examples and Open questions	17
References	19

1. INTRODUCTION

Let X be a non-degenerate reduced, connected closed subscheme in a projective space \mathbb{P}^N defined over an algebraically closed field k and $R = k[x_0, \dots, x_N]$ be the coordinate ring of \mathbb{P}^N . Let $\pi_\Lambda : X \subset \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-t-1}$ denote the projection of X from a linear space $\Lambda = \mathbb{P}^t$. We call it either *outer* projection which is an affine map if

The second author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(grant No.2009-0063180).

$X \cap \Lambda = \emptyset$, projection from a *subscheme* $X \cap \Lambda$ when $X \cap \Lambda \neq \emptyset$ or *inner* projection in case $\Lambda \subset X$. These projections as well as blow-ups have been very useful projective techniques in algebraic geometry. In particular, the inner projection has been as standard issue since del Pezzo and Fano used this projection for the classification of del Pezzo surfaces and Fano 3-folds ([25]).

There are also some classical results about non-birational loci of these projection morphisms and geometric structure of the projection images ([4],[7],[26],[27]).

Besides, from the syzygetic viewpoints, we have found it very interesting to understand some relations between the syzygies of X and its projections as we move the center of projection. In this paper, we develop a method to approach these relations via the elimination mapping cone construction and induced long exact sequence about the graded Tor modules.

Note that the syzygy structures of schemes and their geometric implications have been recently focused when projective schemes have property $\mathbf{N}_{2,p}$ ([8],[10],[11],[12],[17]). This kind of question is also closely related to the Eisenbud-Goto conjecture on Castelnuovo-Mumford regularity and other conjectures on linear syzygies in classical algebraic geometry.

Problems. We list our main problems:

- (a) (Property $\mathbf{N}_{2,p}$ under inner projections) Let X be a nondegenerate reduced, connected scheme in \mathbb{P}^N satisfying property $\mathbf{N}_{2,p}$, $p \geq 1$ which is not necessarily linearly normal. Consider the inner projection from a linear subvariety Λ of $\dim \Lambda = t < p$ with $\Lambda \cap X \neq \emptyset$ and $X_\Lambda = \overline{\pi_\Lambda(X \setminus \Lambda)}$ in \mathbb{P}^{N-t-1} . Is it true that X_Λ satisfies $\mathbf{N}_{2,p-t-1}$? ([1])
- (b) (Embedded linear syzygies) Eisenbud, Green, Hulek and Popescu showed that under some $\mathbf{N}_{2,p}$ -assumption, the syzygies of X restrict surjectively to the syzygies of linear sections in their ‘Restricting linear syzygies’([10]). How do the syzygies behave under inner projections? Is there any natural projection-analogue of the linear section case?
- (c) (Necessary lower bound of quadrics) For $\mathbf{N}_{2,p}$ -variety X , the quadratic defining equations are very essential to decide the properties of X . Especially, between the number of quadrics and higher linear syzygies, it is (roughly) believed that the more quadrics X has, the further higher linear syzygies go. Therefore one can ask ‘how many quadrics does X require to satisfy property $\mathbf{N}_{2,p}$?’ This is a natural question, but not yet so well-known.

In this paper, we mainly intend to solve those problems by using the graded mapping cone construction and the partial elimination ideal theory due to M. Green. We’d like to understand syzygy structures of *infinitely* generated graded modules and to apply them to inner projections.

From now on, let us recall the definitions, preliminaries and convention. One can define that X satisfies property $\mathbf{N}_{2,p}$ (cf. [10]) if the coordinate ring R/I_X (or I_X) have only linear syzygies up to p -th steps. Note that property $\mathbf{N}_{2,p}$ is the same as property \mathbf{N}_p defined by Green-Lazarsfeld if X is a projectively normal embedding.

Let’s review the preliminaries about an inner projection.

- Let $X \subset \mathbb{P}^N = \mathbb{P}(V)$ be a non-degenerate projective variety embedded by a subsystem V of a very ample line bundle \mathcal{L} on X and let $\sigma : \tilde{X} \rightarrow X$ be a blowing up of X at a smooth point $q \in X$. One has the regular morphism $\pi' : \tilde{X} \rightarrow X_q = \overline{\pi_q(X \setminus \{q\})} \subset \mathbb{P}^{N-1} = \mathbb{P}(W)$ given by the subsystem $W := V \cap H^0(X, \mathcal{L} \otimes m_q)$. Then we have the following diagram;

$$\begin{array}{ccc} & \tilde{X} & \\ \sigma \swarrow & & \searrow \pi' \\ X \subset \mathbb{P}^N = \mathbb{P}(V) & \xrightarrow{\pi_q} & X_q \subset \mathbb{P}^{N-1} = \mathbb{P}(W) \end{array}$$

One can say that X admits an inner projection if π' is an embedding for some point $q \in X$. This is equivalent to $q \in X \setminus \text{Trisec}(X)$ where $\text{Trisec}(X)$ is the union of all proper trisecant lines or lines in X . For an embedding π' , the exceptional divisor E is linearly embedded via π' in \mathbb{P}^{N-1} , i.e. $\pi'(E) = \mathbb{P}^{r-1} \subset \mathbb{P}^{N-1}$, $r = \dim(X)$ ([3], [14]).

- Let $R = k[x_0, \dots, x_N] = \text{Sym}(V)$ and $S = k[x_1, x_2, \dots, x_N] = \text{Sym}(W)$ be two polynomial rings where $W \subset V$. Assume $q = (1, 0, \dots, 0) \in X$ (by suitable coordinate change). Then X_q is defined by $I_{X_q} := I_X \cap S$ ideal-theoretically. Note that R/I_X is not finitely generated S -module in case of an inner projection. (Since $q \in X$, there is no polynomial of the form $f = x_0^n + (\text{other terms})$ for some n in I_X). In general, I_X has the following S -module syzygies as infinitely generated :

$$\cdots \rightarrow \bigoplus_{j=2}^{\infty} S(-i-j)^{\beta_{i,j}} \rightarrow \cdots \rightarrow \bigoplus_{j=2}^{\infty} S(-j)^{\beta_{0,j}} \rightarrow I_X \rightarrow 0.$$

From the following basic short exact sequence as S -modules

$$0 \longrightarrow I_{X_q} \longrightarrow I_X \longrightarrow I_X/I_{X_q} \longrightarrow 0,$$

we will show that they have interesting syzygy structures as S -modules even if they are not finitely generated.

- (Scheme-theoretic $\mathbf{N}_{2,p}$) We call that X satisfies scheme-theoretic $\mathbf{N}_{2,p}$ property if there exists a homogeneous ideal I such that its sheafification $\tilde{I} = \mathcal{I}_X$ and R/I satisfies property $\mathbf{N}_{2,p}$. So, if X is scheme-theoretically cut out by quadrics (i.e. satisfies scheme-theoretic $\mathbf{N}_{2,1}$), then we can write I as

$$(1.1) \quad I = (x_0 \ell_1 - Q_{0,1}, \dots, x_0 \ell_t - Q_{0,t}, Q_1, \dots, Q_s), \quad q = (1, 0, \dots, 0) \in X$$

where ℓ_i is a linear form, $Q_{0,i}, Q_j$ are quadratic forms in $S = k[x_1, \dots, x_N]$ and they are minimal generators. We can also assume all $\{\ell_i\}$ are linearly independent, and all $\{Q_{0,i}\}$ are distinct. Then we can say that $\{\ell_i\}$ generate $(T_q X)^*$. Note that $e = \text{codim}(X) = N - r$ if q is a smooth point. In general, e is equal to $N - \dim T_q X$.

The paper is organized as follows. First of all, in Section 2, we set up a graded mapping cone construction for infinitely generated graded modules and develop the partial elimination ideal theory initiated by M. Green so that we can understand the syzygy structures of infinitely generated graded modules. This partial elimination ideal theory gives us a *local* information of X at q , which turns out to govern syzygies and other properties of the inner projection X_q from the (*global*) homogeneous equations.

In Section 3, we obtain some results on syzygy structures and geometric properties of *inner* projections, i.e. embedded linear syzygies, the number of quadratic equations, and the arithmetic depth comparing all these results with the outer projection cases ([24]). In particular, we can show that the inner projection from any smooth point of X satisfies at least property $\mathbf{N}_{2,p-1}$ for a projective reduced connected scheme X of codimension e satisfying property $\mathbf{N}_{2,p}$, $p \geq 1$ and the arithmetic depths of inner projections are equal to that of the original variety if it is cut out by quadrics. These results look very interesting enough to lead to the “rigidity theorem” on property $\mathbf{N}_{2,p}$, $e-1 \leq p \leq e$ as classifications. We also give the sharp lower bound $e \cdot p - \frac{p(p-1)}{2}$ of the number for quadrics vanishing on X satisfying property $\mathbf{N}_{2,p}$, $p \geq 1$. Finally, in Section 4 we give some interesting examples and open questions for the subsequent work.

Convention. We are working on the following convention:

- Let $X \subset \mathbb{P}^N$ be nondegenerate reduced, connected closed subscheme defined over an algebraically closed field k of $\dim X = r$ and $q \in X$ be a smooth point $(1 : 0 : \cdots : 0)$ by coordinate change.
- We abbreviate $\mathrm{Tor}_i^R(M, k)_{i+j}$ as $\mathrm{Tor}_i^R(M)_{i+j}$ (same for S -module Tor).
- (Betti numbers) We remind readers that $\mathrm{Tor}_i^R(R/I)_{i+j} = \mathrm{Tor}_{i-1}^R(I)_{i-1+j+1}$. So $\beta_{i,j}^R(R/I) = \beta_{i-1,j+1}^R(I)$. We often write $\beta_{i,j}^R(I)$ as $\beta_{i,j}(I)$ or $\beta_{i,j}$ if it is obvious.
- (Arithmetic depth) We mean the arithmetic depth of X , $\mathrm{depth}_R(R/I_X)$ when we refer the depth of X and denote by $\mathrm{depth}_R(X)$.

Acknowledgements The first author would like to thank Professor Frank-Olaf Schreyer for hosting his visit to Saarbrücken under KOSEF-DAAD Summer Institute Program and for many valuable comments preparing this paper. The second author would like to thank Professor B. Sturmfels, M. Brodmann for their useful comments, especially P. Schenzel for valuable discussion and Example 4.1 during their stay in Korea Institute of Advanced Study(KIAS) and KAIST, Korea in the Summer 2009. We would also like to thank Professor F.Zak who informed us of Professors A.Alzati and J.Sierra’s recent preprint related to our paper (Remark 3.9).

2. GRADED MAPPING CONE CONSTRUCTION AND PARTIAL ELIMINATION IDEALS

In general the mapping cone of the chain map between two complexes is a kind of natural extension of complexes induced by the given chain map. We will construct some “graded” mapping cone which is related to projections and is useful to understand the syzygies of projections. Mainly we will exploit this construction to study a simple *inner* projection. Another ingredient is the partial elimination ideal theory. Let us construct the graded mapping cone theorem and review the partial elimination ideals briefly.

A Graded Mapping Cone Construction. Let $W = k\langle x_1, \dots, x_N \rangle \subset V = k\langle x_0, \dots, x_N \rangle$ be vector spaces over k and $S = \mathrm{Sym}(W) = k[x_1, \dots, x_N] \subset R = \mathrm{Sym}(V) = k[x_0, \dots, x_N]$ be polynomial rings.

- M : a graded R -module given a degree 1 shifting map by μ
(i.e. $\mu : M_i \rightarrow M_{i+1}$)

- \mathbb{G}_* (resp. \mathbb{F}_*) : the graded Koszul complex of M , $K_*^S(M)$ (resp. $M[-1]$, $K_*^S(M[-1])$) as follows:

$$0 \rightarrow \wedge^N W \otimes M \rightarrow \cdots \rightarrow \wedge^2 W \otimes M \rightarrow W \otimes M \rightarrow M \rightarrow 0$$

whose graded components $(\mathbb{G}_i)_{i+j}$ are $K_i^S(M)_{i+j} = \wedge^i W \otimes M_j$ (resp. $(\mathbb{F}_i)_{i+j} = \wedge^i W \otimes M_{j-1}$).

- Then $\mu : M_i \rightarrow M_{i+1}$ induces the chain map $\bar{\mu} : \mathbb{F}_* = K_*^S(M[-1]) \rightarrow \mathbb{G}_* = K_*^S(M)$ of degree 0.

Now we construct the mapping cone $(\mathbb{C}_{\bar{\mu}}, d_{\bar{\mu}})$ such that

$$(2.1) \quad 0 \rightarrow \mathbb{G}_* \rightarrow (\mathbb{C}_{\bar{\mu}})_* \rightarrow \mathbb{F}_*[-1] \rightarrow 0,$$

where $\mathbb{C}_{\bar{\mu}}$ is a direct sum $\mathbb{G}_* \oplus \mathbb{F}_*[-1]$ and the differential $d_{\bar{\mu}}$ is given by

$$(d_{\bar{\mu}})_* = \begin{pmatrix} \partial_{\mathbb{G}} & (-1)^{*+1} \bar{\mu} \\ 0 & \partial_{\mathbb{F}} \end{pmatrix},$$

where ∂ is the differential of Koszul complex. From the construction, it can be checked that we have the following isomorphism([1]):

$$\mathrm{Tor}_i^R(M)_{i+j} \simeq H_i((\mathbb{C}_{\bar{\mu}})_*)_{i+j}.$$

Suppose M is a graded R -module which is also a graded S -module. Consider a multiplication map $\mu = \times x_0$ as a naturally given degree 1 shifting map on M . In this case, the long exact sequence on homology groups induced from (2.1) is important and very useful to study the syzygies of projections algebraically.

Theorem 2.1. (Mapping cone sequence)

Let $S = k[x_1, \dots, x_N] \subset R = k[x_0, x_1, \dots, x_N]$ be two polynomial rings.

- (a) Let M be a graded R -module which is not necessarily finitely generated. Then, we have a natural long exact sequence:

$$\cdots \rightarrow \mathrm{Tor}_i^R(M)_{i+j} \rightarrow \mathrm{Tor}_{i-1}^S(M)_{i-1+j} \xrightarrow{\bar{\mu}} \mathrm{Tor}_{i-1}^S(M)_{i-1+j+1} \rightarrow \mathrm{Tor}_{i-1}^R(M)_{i-1+j+1} \cdots$$

whose connecting homomorphism $\bar{\mu}$ is induced by the multiplicative map $\times x_0$.

- (b) Assume that R/I satisfies property $\mathbf{N}_{d,p}$ for some $d \geq 2, p \geq 1$. Then a multiplication by x_0 induces a sequence of isomorphisms on $\mathrm{Tor}_i^S(I)_{i+j}$ for $0 \leq i \leq p-2, j \geq d+1$ and a surjection for $j = d$;

$$\cdots \xrightarrow{\times x_0} \mathrm{Tor}_i^S(I)_{i+d} \xrightarrow{\times x_0} \mathrm{Tor}_i^S(I)_{i+d+1} \xrightarrow{\times x_0} \mathrm{Tor}_i^S(I)_{i+d+2} \xrightarrow{\times x_0} \cdots$$

For $i = p-1$, we have a sequence of surjections from $j = d$

$$\cdots \xrightarrow{\times x_0} \mathrm{Tor}_{p-1}^S(I)_{p-1+d} \xrightarrow{\times x_0} \mathrm{Tor}_{p-1}^S(I)_{p-1+d+1} \xrightarrow{\times x_0} \mathrm{Tor}_{p-1}^S(I)_{p-1+d+2} \xrightarrow{\times x_0} \cdots$$

Remark 2.2. J. Ahn and the second author pointed out that this graded mapping cone construction is closely related to outer projections ([1]). We remark here that this theorem is also true even for an *infinitely generated* S -module M and relates the torsion module $\mathrm{Tor}^R(M)$ to the torsion module of M as S -module. Therefore this gives us useful information about syzygies of *inner* projections.

Proof. (a) follows from theorem 2.2 in [1]. For a proof of (b), consider the mapping cone sequence of Theorem (2.1) for $M = I$

$$\mathrm{Tor}_{i+1}^R(I)_{i+1+j} \rightarrow \mathrm{Tor}_i^S(I)_{i+j} \xrightarrow{\times x_0} \mathrm{Tor}_i^S(I)_{i+j+1} \rightarrow \mathrm{Tor}_i^R(I)_{i+j+1}$$

Note that $\mathrm{Tor}_i^R(I, k)_{i+j} = 0$ for $0 \leq i \leq p-1$ and $j \geq d+1$ by assumption that I has $\mathbf{N}_{d,p}$ property as a R -module. So, we have an isomorphism

$$\mathrm{Tor}_i^S(I, k)_{i+j} \xrightarrow{\times x_0} \mathrm{Tor}_i^S(I, k)_{i+j+1}$$

for $0 \leq i \leq p-2$, $\forall j \geq d+1$ and a surjection for $j = d$.

In case $i = p-1$, we know $\mathrm{Tor}_{p-1}^R(I)_{p-1+j} = 0$ for $j \geq d+1$ in the mapping cone sequence

$$\mathrm{Tor}_p^R(I)_{p+j} \rightarrow \mathrm{Tor}_{p-1}^S(I)_{p-1+j} \xrightarrow{\times x_0} \mathrm{Tor}_{p-1}^S(I)_{p-1+j+1} \rightarrow \mathrm{Tor}_{p-1}^R(I)_{p-1+j+1}.$$

Therefore we get the desired surjections for $i = p-1$ case. \square

Partial Elimination Ideals under a Projection. Mark Green introduced partial elimination ideals in his lecture note [16]. For the degree lexicographic order, if $f \in I_m$ has leading term $\mathrm{in}(f) = x_0^{d_0} \cdots x_n^{d_n}$, we set $d_0(f) = d_0$, the leading power of x_0 in f . Then we can give the definition of partial elimination ideals as in the following.

Definition 2.3. Let $I \subset R$ be a homogeneous ideal and let

$$\tilde{K}_i(I) = \bigoplus_{m \geq 0} \{f \in I_m \mid d_0(f) \leq i\}.$$

If $f \in \tilde{K}_i(I)$, we may write uniquely $f = x_0^i \bar{f} + g$ where $d_0(g) < i$. Now we define the ideal $K_i(I)$ in S generated by the image of $\tilde{K}_i(I)$ under the map $f \mapsto \bar{f}$ and we call $K_i(I)$ the i -th partial elimination ideal of I .

Observation 2.4. We can observe some properties of these ideals in the projection case.

- (a) 0-th partial elimination ideal $K_0(I_X)$ of I_X is

$$I_{X_q} = \bigoplus_{m \geq 0} \{f \in (I_X)_m \mid d_0(f) = 0\} = I_X \cap S.$$

- (b) $\tilde{K}_i(I_X)$ is a natural filtration of I_X with respect to x_0 , so we have the following filtration:

$$\tilde{K}_0(I_X) \subset \tilde{K}_1(I_X) \subset \cdots \subset \tilde{K}_i(I_X) \subset \cdots \subset \tilde{K}_\infty(I_X) = I_X$$

$$I_{X_q} = K_0(I_X) \subset K_1(I_X) \subset \cdots \subset K_i(I_X) \subset K_{i+1}(I_X) \subset \cdots \subset S.$$

- (c) $\tilde{K}_i(I_X)$ is a finitely generated graded S -module and there is a short exact sequence as graded S -modules

$$(2.2) \quad 0 \rightarrow \frac{\tilde{K}_{i-1}(I_X)}{\tilde{K}_0(I_X)} \rightarrow \frac{\tilde{K}_i(I_X)}{\tilde{K}_0(I_X)} \rightarrow K_i(I_X)(-i) \rightarrow 0.$$

In general we can see when the $K_i(I_X)$'s stabilize and what it looks like for inner projections. From the homogeneous defining equations, this partial elimination ideal gives us a local information of X at q and in next section it will turn out that the information govern syzygies and other properties of X_q . The following proposition is very useful to understand the defining equations and syzygies of inner projections.

Proposition 2.5. Let $X \subset \mathbb{P}^N$ be a reduced and connected projective scheme with the saturated homogeneous ideal I_X . Let $q = (1, 0, \dots, 0) \in X$.

- (a) If I_X satisfies property $\mathbf{N}_{d,1}$, $K_i(I_X)$ stabilizes at least at $i = d-1$ to an ideal defining TC_qX , the tangent cone of X at q . So if q is smooth, $K_{d-1}(I_X)$ consists of linear forms which defines T_qX .
- (b) In particular, if I_X is generated by quadrics and q is smooth, then $K_i(I_X)$ stabilizes at $i = 1$ step to an ideal $I_L = (l_1, \dots, l_e)$, $e = \text{codim}(X, \mathbb{P}^N)$ which defines the tangent space T_qX , i.e. $I_{X_q} \subset I_L = K_1(I_X) = \dots = K_i(I_X) = \dots \subset S$ and I_X/I_{X_q} has obvious syzygies as an infinitely generated S -module such that:

$$\begin{array}{ccccccc}
& S(-e-1)^{b_e} & & S(-3)^{b_2} & & S(-2)^{b_1} & \\
0 \rightarrow & \oplus S(-e-2)^{b_e} & \rightarrow \dots \rightarrow & \oplus S(-4)^{b_2} & \rightarrow & \oplus S(-3)^{b_1} & \rightarrow I_X/I_{X_q} \rightarrow 0, \\
& \oplus S(-e-3)^{b_e} & & \oplus S(-5)^{b_2} & & \oplus S(-4)^{b_1} & \\
& \dots & & \dots & & \dots &
\end{array}$$

where $b_i = \binom{e}{i}$.

Proof. (a) Since I_X is generated in $\deg \leq d$ and $q = (1, 0, \dots, 0) \in X$, we have generators $\{F_i\}$ of I_X with $d_0(F_i) \leq d-1$ because there is no generator of the form $x_0^d + \text{other lower terms in } x_0$. From this, every leading term f of a homogeneous polynomial F in I_X of $\deg k$ ($k \geq d$) is written as $x_0^{\square} \cdot \bar{f}$ where $\bar{f} \in K_c(I_X)$ for some $c \leq d-1$. So $K_i(I_X)$ stabilizes at least at $i = d-1$. Note that all $\bar{f} \in K_i(I_X)$ ($i \geq 0$) are also regarded as the defining equations of tangent cone TC_qX of X at q because they come from $f = x_0^i \bar{f} + g \in I_X$, $d_0(g) < i$. Therefore, $K_i(I_X)$ stabilizes to the ideal defining TC_qX . In case of a smooth point $q \in X$, $T_qX = TC_qX$ and $K_{d-1}(I_X) = (\ell_1, \dots, \ell_e)$, $e = \text{codim}(X, \mathbb{P}^N)$.

(b) Since $d = 2$ and q is a smooth point, $K_i(I_X)$ becomes $I_L = (\ell_1, \dots, \ell_e)$ for each $i \geq 1$. For the sake of the S -module syzygy of I_X/I_{X_q} , first note that $I_X = \tilde{K}_{\infty}(I_X)$. From the exact sequence (2.2), we get $\tilde{K}_1(I_X)/I_{X_q} \simeq K_1(I_X)(-1)$ with the following linear Koszul resolution: letting $b_i = \binom{e}{i}$,

$$0 \rightarrow S(-e-1)^{b_e} \rightarrow \dots \rightarrow S(-3)^{b_2} \rightarrow S(-2)^{b_1} \rightarrow \tilde{K}_1(I_X)/I_{X_q} \rightarrow 0.$$

Next, $K_2(I_X)(-2) = K_1(I_X)(-2)$ has also linear syzygies:

$$0 \rightarrow S(-e-2)^{b_e} \rightarrow \dots \rightarrow S(-4)^{b_2} \rightarrow S(-3)^{b_1} \rightarrow K_2(I_X)(-2) \rightarrow 0,$$

and we have the following exact sequence from (2.2) again,

$$0 \rightarrow \frac{\tilde{K}_1(I_X)}{I_{X_q}} \rightarrow \frac{\tilde{K}_2(I_X)}{I_{X_q}} \rightarrow K_2(I_X)(-2) \rightarrow 0.$$

By the long exact sequence of Tor, we know that

$$\begin{array}{ccccccc}
& S(-e-1)^{b_e} & & S(-3)^{b_2} & & S(-2)^{b_1} & \\
0 \rightarrow & \oplus S(-e-2)^{b_e} & \rightarrow \dots \rightarrow & \oplus S(-4)^{b_2} & \rightarrow & \oplus S(-3)^{b_1} & \rightarrow \tilde{K}_2(I_X)/I_{X_q} \rightarrow 0.
\end{array}$$

Similarly, we can compute the syzygy of $\tilde{K}_i(I_X)/I_{X_q}$ for any i , and we get the desired resolution of $I_X/I_{X_q} = \tilde{K}_{\infty}(I_X)/I_{X_q}$ as S -module in the end. \square

Remark 2.6. (Outer projection case)

- (a) We can also consider outer projection by the similar method. In this case $K_i(I_X)$ always stabilizes at least at d -th step to $(1) = S$ if I_X satisfies $\mathbf{N}_{d,1}$. More interesting fact is that $K_{d-1}(I_X)$ consists of linear forms with $\mathbf{N}_{d,2}$ -condition. Especially, suppose that X satisfy property $\mathbf{N}_{2,2}$ and $q =$

$(1, 0, \dots, 0) \notin X$. Then $K_1(I_X)$ is an ideal of linear forms I_Σ defining the singular locus Σ of π_q in $X_q \subset \mathbb{P}^{N-1}$ (see [1] for details). By the similar method as in the inner projection, we see that I_X/I_{X_q} has simple S -module syzygies such that:

$$0 \rightarrow S(-t-1)^{b_t} \rightarrow \dots \rightarrow S(-3)^{b_2} \rightarrow S(-2)^{b_1+1} \rightarrow I_X/I_{X_q} \rightarrow 0,$$

$$\quad \quad \quad \oplus S(-3)$$

$$\quad \quad \quad \oplus S(-4)$$

$$\quad \quad \quad \dots$$

where $b_i = \binom{t}{i}$, $t = \text{codim}(\Sigma, \mathbb{P}^{N-1})$. So, this resolution can be used to study the outer projection case.

- (b) The stabilized ideal gives an important information for projections. In outer case of $\mathbf{N}_{2,p}$ ($p \geq 2$), it is shown in ([1],[24]) that the dimension of Σ determines the number of quadric equations and the arithmetic depth of projected varieties according to moving the center of projection. In our inner projection, $K_1(I_X)$ also shows us the tangential behavior of X at q and $TC_q X$ plays an important role in our problem.

Now some natural questions arise at this point. How are the syzygies of I_{X_q} related to the S -module syzygies of I_X and to the R -module syzygies of I_X ? Specifically, we may ask the following under some $\mathbf{N}_{2,p}$ assumption: Is I_{X_q} generated only by quadrics if so I_X is? There might be cubic generators like $\ell_i Q_{0,j} - \ell_j Q_{0,i} (= \ell_j \cdot [x_0 \ell_i - Q_{0,i}] - \ell_i \cdot [x_0 \ell_j - Q_{0,j}])$ in I_{X_q} (see (1.1)). If not, how about the case of $\mathbf{N}_{2,2}$? What can we say about higher linear syzygies of X_q ? We will answer these kind of syzygy and elimination problems and derive stronger results by using the graded mapping cone sequence and the partial elimination ideals $K_i(I_X)$ in next section.

Remark 2.7. (Commutativity of x_0 -multiplication) Consider the S -module homomorphism $\phi : I_X \rightarrow I_X/I_{X_q}$, the natural quotient map and also consider multiplication maps in both I_X and I_X/I_{X_q} . This multiplication $\times x_0$ is not well-defined in I_X/I_{X_q} , while it is a well-defined S -module homomorphism in I_X . But, if X is cut by quadrics and q is a smooth point, then the multiplication $\times x_0$ is well-defined in $\text{Tor}^S(I_X/I_{X_q})$ and we have a commutative diagram in the Tor level as follows:

$$\begin{array}{ccc} \text{Tor}_i^S(I_X)_{i+j+1} & \xrightarrow{\phi} & \text{Tor}_i^S(I_X/I_{X_q})_{i+j+1} \\ \uparrow \times x_0 & & \uparrow \times x_0 \\ \text{Tor}_i^S(I_X)_{i+j} & \xrightarrow{\phi} & \text{Tor}_i^S(I_X/I_{X_q})_{i+j} \end{array}$$

, because Proposition 2.5 (b) tells us that the syzygies of I_X/I_{X_q} essentially come from the syzygies of *leading terms in x_0 -elimination order*, nothing but the Koszul syzygies of $\{x_0^\alpha \ell_1, \dots, x_0^\alpha \ell_e\}$. Therefore the x_0 -multiplication makes above diagram commuting. For more details and general conditions for this, see the Remark 2.6 (d), (e) in [18].

Remark 2.8. (Scheme-theoretic $\mathbf{N}_{2,p}$ -case) Since our method is available for not only the homogeneous ideal I_X , but also any ideal I cutting X scheme-theoretically. Thus all the results in this paper also hold for the *scheme-theoretic* $\mathbf{N}_{2,p}$ -case with suitable corresponding modifications.

3. EMBEDDED LINEAR SYZYGIES, DEPTH AND APPLICATIONS

For a projective variety X , property $\mathbf{N}_{2,p}$ is a natural generalization of property \mathbf{N}_p due to M. Green. The linear syzygies of a given embedded variety give strong geometric information of the variety, mainly related the higher secant variety. Recently, Eisenbud, Green, Hulek, Popescu showed that under some $\mathbf{N}_{2,p}$ ($p \geq 1$) assumption, these syzygies restrict surjectively to linear sections in their ‘Restricting linear syzygies’, [10]. We consider in this section the behavior of the syzygies under inner projections and we present our main theorem on ‘embedded linear syzygies’ which is the natural projection-analogue of the linear section case.

Theorem 3.1. *Let $X \subset \mathbb{P}^N$ be a non-degenerate reduced and connected projective scheme satisfying property $\mathbf{N}_{2,p}$ for some $p \geq 1$ and $q \in X$ be a smooth point. Consider the inner projection $\pi_q : X \dashrightarrow X_q \subset \mathbb{P}^{N-1}$. Then there is an injection between the minimal free resolutions of I_{X_q} and I_X up to first $(p-1)$ -th step, i.e.*

$$\exists f : \mathrm{Tor}_i^S(I_{X_q}, k)_{i+j} \hookrightarrow \mathrm{Tor}_i^R(I_X, k)_{i+j} \text{ for } 0 \leq i \leq p-2, \forall j \in \mathbb{Z}$$

which is induced by the natural inclusion $I_{X_q} \hookrightarrow I_X$ and the mapping cone sequence.

Proof. We have a basic short exact sequence of S -modules,

$$(3.1) \quad 0 \longrightarrow I_{X_q} \longrightarrow I_X \longrightarrow I_X/I_{X_q} \longrightarrow 0.$$

From the long exact sequence of (3.1) and the mapping cone sequence of I_X in (2.1), we have a diagram

$$\begin{array}{ccccc} 0 & & 0 & & \\ \downarrow & & \downarrow & & \\ \mathrm{Tor}_i^S(I_{X_q}, k)_{i+j-1} & & \mathrm{Tor}_i^S(I_{X_q}, k)_{i+j} & & \\ \downarrow & & \downarrow g & \searrow f := h \circ g & \\ \mathrm{Tor}_i^S(I_X, k)_{i+j-1} & \xrightarrow{\times x_0} & \mathrm{Tor}_i^S(I_X, k)_{i+j} & \xrightarrow{h} & \mathrm{Tor}_i^R(I_X, k)_{i+j} \\ \downarrow & & \downarrow & & \\ \mathrm{Tor}_i^S(I_X/I_{X_q}, k)_{i+j-1} & \xrightarrow{\times x_0} & \mathrm{Tor}_i^S(I_X/I_{X_q}, k)_{i+j} & & \end{array}$$

For any $0 \leq i \leq p-2$, we proceed with j case by case.

Case 1) $j \leq 1$: Since $\mathrm{Tor}_i^S(I_{X_q}, k)_{i+j} = 0$, so it is obviously injected to $\mathrm{Tor}_i^R(I_X, k)_{i+j}$ by f .

Case 2) $j = 2$ (i.e. linear syzygy cases for each i): From (3.1), we have

$$\mathrm{Tor}_{i+1}^S(I_X/I_{X_q}, k)_{i+2} \longrightarrow \mathrm{Tor}_i^S(I_{X_q}, k)_{i+2} \xrightarrow{g} \mathrm{Tor}_i^S(I_X, k)_{i+2}$$

Since q is a smooth point, with $N_{2,1}$ condition we know the syzygy structures of I_X/I_{X_q} as S -module by Proposition 2.5 (b). It shows $\mathrm{Tor}_{i+1}^S(I_X/I_{X_q}, k)_{i+2} = 0$,

implying that g is injective. Since X is non-degenerate, so $\mathrm{Tor}_i^S(I_X, k)_{i+1} = 0$ and h is also injective at the horizontal mapping cone sequence of above diagram. Hence f is injective in this case, too.

Case 3) $j \geq 3$: First note that $\mathrm{Tor}_i^R(I_X, k)_{i+j} = 0$ for $0 \leq i \leq p-1, j \geq 3$ by the assumption of property $\mathbf{N}_{2,p}$. We will show that g is injective and $\mathrm{Tor}_i^S(I_X, k)_{i+j}$ is isomorphic to $\mathrm{Tor}_i^S(I_X/I_{X_q}, k)_{i+j}$ for $0 \leq i \leq p-2$. Then we can conclude that $\mathrm{Tor}_i^S(I_{X_q}, k)_{i+j} = 0$, so f is injective for $0 \leq i \leq p-2$.

Consider the commutative diagram (by Remark 2.7) in the third quadrant part of above diagram. Repeating this diagram by multiplying x_0 sufficiently, we have the following diagram:

$$\begin{array}{ccccc} \mathrm{Tor}_{i+1}^S(I_X, k)_{i+1+j-1} & \xrightarrow[\text{surj.}]{\cdot} & \mathrm{Tor}_{i+1}^S(I_X/I_{X_q}, k)_{i+1+j-1} & \longrightarrow & \mathrm{Tor}_i^S(I_{X_q}, k)_{i+j} \xrightarrow{g} \\ \downarrow \text{surj.} & & \downarrow \text{isom.} & & \\ \mathrm{Tor}_{i+1}^S(I_X, k)_{i+1+N} & \xrightarrow{\text{isom.}} & \mathrm{Tor}_{i+1}^S(I_X/I_{X_q}, k)_{i+1+N} & & \end{array}$$

The left vertical map is surjective from Theorem 2.1 (b), and the right one is an isomorphism by the syzygy structures of I_X/I_{X_q} in Proposition 2.5 (b). Since I_{X_q} is a finite S -module, $\mathrm{Tor}_i^S(I_{X_q}, k)_{i+N} = 0$ for sufficiently large N , so we get the below(second row) isomorphism. Therefore the map

$$\mathrm{Tor}_{i+1}^S(I_X, k)_{i+1+j-1} \rightarrow \mathrm{Tor}_{i+1}^S(I_X/I_{X_q}, k)_{i+1+j-1}$$

is surjective, and g is injective.

Similarly, we can have the desired isomorphism between $\mathrm{Tor}_i^S(I_X, k)_{i+j}$ and $\mathrm{Tor}_i^S(I_X/I_{X_q}, k)_{i+j}$ as follows:

$$\begin{array}{ccccc} \mathrm{Tor}_i^S(I_{X_q}, k)_{i+j} & \xrightarrow{g} & \mathrm{Tor}_i^S(I_X, k)_{i+j} & \xrightarrow[\text{isom.}]{\alpha} & \mathrm{Tor}_i^S(I_X/I_{X_q}, k)_{i+j} \\ & & \downarrow \text{isom.} & & \downarrow \text{isom.} \\ & & \mathrm{Tor}_i^S(I_X, k)_{i+N} & \xrightarrow{\text{isom.}} & \mathrm{Tor}_i^S(I_X/I_{X_q}, k)_{i+N} \end{array}$$

In this case, the mapping cone construction gives the left vertical isomorphism by Theorem 2.1 (b). So the above map α is an isomorphism as we wish, and $\mathrm{Tor}_i^S(I_{X_q}, k)_{i+j} = 0$ for $0 \leq i \leq p-2, j \geq 3$. \square

This main Theorem 3.1 tell us that all the S -module syzygies of X_q are exactly the very ones which are *already* embedded in the linear syzygies of X as R -module. This doesn't hold for outer projection and inner projection of varieties with $\mathbf{N}_{d,p}$ ($d \geq 3$).

Example 3.2. Let C be a rational normal curve in \mathbb{P}^3 and I_C be the homogeneous ideal $(x_0x_2 - x_1^2, x_0x_1 - x_1x_3 - x_2^2, x_0^2 - x_0x_3 - x_1x_2)$ under suitable coordinate change. We know C is 2-regular and consider an outer projection of C from $q = (1, 0, 0, 0)$. Then $I_{C_q} = (x_1^3 - x_1x_2x_3 - x_2^3)$ has a cubic generator (i.e. $\mathbf{N}_{3,1}$). Since $x_1^3 - x_1x_2x_3 - x_2^3 = (-x_1) \cdot [x_0x_2 - x_1^2] + x_2 \cdot [x_0x_1 - x_1x_3 - x_2^2]$, this is zero in $\mathrm{Tor}_0^R(I_C)_3$ and $\mathrm{Tor}_0^S(I_{C_q})_3 \rightarrow \mathrm{Tor}_0^R(I_C)_3$ is not injective. In general, if we take the

center $q \in L \simeq \mathbb{P}^2$ which is a multiseccant 2-plane, then for outer and inner projection cases there is a multiseccant line to X_q . So, the defining equations of X_q may have larger degrees.

As an immediate consequence, we have the following corollary.

Corollary 3.3. (Property $\mathbf{N}_{2,p-1}$ of inner projections) *With the same hypothesis as Theorem 3.1, the projected scheme X_q is cut out by quadrics and satisfies property $\mathbf{N}_{2,p-1}$.*

Proof. **Case 3)** $j \geq 3$ in the proof of Theorem 3.1 is a proof of the corollary. \square

Remark 3.4. For a smooth irreducible variety $X \subset \mathbb{P}(H^0(\mathcal{L}))$ with the condition \mathbf{N}_p ($p \geq 1$) embedded by the complete linear system of a very ample line bundle \mathcal{L} on X , Y. Choi, P. Kang and S. Kwak showed that the inner projection X_q is smooth and satisfies property \mathbf{N}_{p-1} for any $q \in X \setminus \text{Trisec}(X)$, i.e. property \mathbf{N}_{p-1} holds for $(\text{Bl}_q(X), \sigma^*\mathcal{L} - E)$ by using vector bundle techniques and Koszul cohomology methods due to Green-Lazarsfeld ([8]). Our Corollary 3.3 extends this result to the category of reduced and connected projective schemes satisfying property $\mathbf{N}_{2,p}$ with any smooth point $q \in X$. Note that this uniform behavior looks unusual in a sense that linear syzygies of outer projections heavily depend on moving the center of projection in an ambient space \mathbb{P}^N ([9],[22], [24]).

In order to understand the Betti diagram of inner projections, we need to consider defining equations of inner projections, depth, and the Castelnuovo-Mumford regularity.

Proposition 3.5. (Quadratic equations of inner projections)

Let $X \subset \mathbb{P}^N$ be a non-degenerate, reduced, and connected scheme satisfying $\mathbf{N}_{2,1}$ and any (possibly singular) $q \in X$. Consider the inner projection $\pi_q : X \dashrightarrow X_q \subset \mathbb{P}^{N-1}$. Then,

$$h^0(\mathbb{P}^{N-1}, \mathcal{I}_{X_q}(2)) = h^0(\mathbb{P}^N, \mathcal{I}_X(2)) - N + \dim T_q X$$

Proof. If q is smooth, then this is an immediate result from the proof of Theorem 3.1. We could observe $\text{Tor}_0^S(I_X, k)_2 = \text{Tor}_0^S(I_{X_q}, k)_2 + \text{Tor}_0^S(I_X/I_{X_q}, k)_2$. Since $\text{Tor}_0^S(I_X, k)_2 = h^0(\mathcal{I}_X(2))$, $\text{Tor}_0^S(I_{X_q}, k)_2 = h^0(\mathcal{I}_{X_q}(2))$ and $\text{Tor}_0^S(I_X/I_{X_q}, k)_2 = t = \text{codim } X$ by Proposition 2.5, we get the formula desired.

If q is singular, then $T_q X \neq TC_q X$. So we may write

$$K_1(I_X) = (\ell_1, \dots, \ell_t, \text{higher degree polynomials})$$

where $I_X = (x_0\ell_1 - Q_{0,1}, \dots, x_0\ell_t - Q_{0,t}, Q_1, \dots, Q_s)$ and $\dim \text{Tor}_0^S(I_X, k)_2 = t + s$. Even though $K_1(I_X)$ does not consist of only linear polynomials but also has higher degree ones, we could consider the degree 2-part of $\text{Tor}_i^S(I_X/I_{X_q}, k)$ in the same manner of a smooth case. That is, we have

$$\begin{cases} \dim \text{Tor}_0^S(K_1(I_X)(-1))_2 = t \ (i = 1) \\ \dim \text{Tor}_0^S(K_i(I_X)(-i))_2 = 0 \ (i \geq 2) : \text{quadratic generators of } K_i(I_X)(-i) \\ \dim \text{Tor}_1^S(K_i(I_X)(-i))_2 = 0 \ (i \geq 1) : \text{1st linear syzygy of } K_i(I_X)(-i), \end{cases}$$

because $K_1(I_X) = K_2(I_X) = \dots$ and $\dim \text{Tor}_0^S(K_1(I_X))_1 = t$. By same reasoning in Proposition 2.5, we get $\dim \text{Tor}_0^S(I_X/I_{X_q})_2 = t$, $\dim \text{Tor}_1^S(I_X/I_{X_q})_2 = 0$.

Therefore from the sequence

$$\rightarrow \mathrm{Tor}_1^S(I_X/I_{X_q})_2 \rightarrow \mathrm{Tor}_0^S(I_{X_q})_2 \rightarrow \mathrm{Tor}_0^S(I_X)_2 \rightarrow \mathrm{Tor}_0^S(I_X/I_{X_q})_2 \rightarrow 0$$

we conclude $h^0(\mathcal{J}_{X_q}(2)) = \dim \mathrm{Tor}_0^S(I_{X_q})_2 = (t+s) - t = h^0(\mathcal{J}_X(2)) - (N - \dim T_q X)$ because $\{\ell_1, \dots, \ell_t\}$ defines $T_q X$. \square

Remark 3.6. In outer projection case, there is a formula $h^0(\mathcal{J}_{X_q}(2)) = h^0(\mathcal{J}_X(2)) - (N - \dim \Sigma_q(X))$ if X satisfies property $\mathbf{N}_{2,2}$ (see Proposition 3.9, [1] and theorem 3.3, [24]). This also shows that there is a tendency of having more quadrics for projected varieties as q is getting closer to X . Note that the negative value of $h^0(\mathcal{J}_{X_q}(2))$ implies that there is no quadric vanishing on X_q . By this fact, we can expect that the inner projection case has more linear syzygies as Corollary 3.3 shows.

Next question is that how many quadrics defining X are required to satisfy property $\mathbf{N}_{2,p}$ and we give the sharp lower bound in the following.

Corollary 3.7. (Lower bound of quadrics for property $\mathbf{N}_{2,p}$) *Let X be a non-degenerate reduced and connected subscheme in \mathbb{P}^{r+e} of codimension e and I be an ideal which defines X scheme-theoretically. Suppose that I satisfies property $\mathbf{N}_{2,p}$ and $\beta_{0,2}(I)$ is the number of quadric generators of I . Then $\beta_{0,2}(I)$ is not less than LB_p as follows:*

$$LB_p = e \cdot p - \frac{p(p-1)}{2} \leq \beta_{0,2}(I) \leq \beta_{0,2}(I_X) (= h^0(\mathcal{J}_X(2)))$$

Proof. Let's take a smooth point q_0 in X and project X from q_0 . Let $X^{(1)}$ be the image (the Zariski closure) and $I^{(1)}$ be the elimination ideal of I . Then, from Proposition 3.5 we get

$$\beta_{0,2}(I^{(1)}) = \beta_{0,2}(I) - (r+e) + r.$$

We also know that $I^{(1)}$ defines $X^{(1)}$ scheme-theoretically and satisfies property $\mathbf{N}_{2,p-1}$. Take another smooth point q_1 in $X^{(1)}$ and project it from q_1 . Then, with the same notation, we have

$$\beta_{0,2}(I^{(2)}) = \beta_{0,2}(I^{(1)}) - (r+e-1) + r.$$

Taking successive inner projections, we get

$$\beta_{0,2}(I^{(p-1)}) = \beta_{0,2}(I^{(p-2)}) - (r+e-p+2) + r.$$

Summing up both sides of above equations, it gives

$$\beta_{0,2}(I^{(p-1)}) = \beta_{0,2}(I) - \frac{(p-1)(2r+2e-p+2)}{2} + r(p-1) \dots (*)$$

And we know that $X^{(p-1)}$ is still cut by quadrics (i.e. $\mathbf{N}_{2,1}$). So $\beta_{0,2}(I^{(p-1)})$ is not less than $\mathrm{codim} X^{(p-1)} = (r+e) - p + 1 - r = e - p + 1$. If we plug-in this inequality to (*), we've got the desired bound LB_p . \square

Remark 3.8. This bound is sharp for $p = 1$ by complete intersections, $p = e - 1$ by next to minimal degree varieties and $p = e$ by minimal degree varieties. Note also that the upper bound for $\beta_{0,2}(I_X)$ for a non-degenerate integral subscheme $X \subset \mathbb{P}^{r+e}$ of codimension e is $\frac{e(e+1)}{2}$ and this maximum number can be attained if and only if the variety X is of minimal degree from Corollaries 5.4, 5.8 in [28].

Remark 3.9. (Degree bound by property $\mathbf{N}_{2,p}$) Recently, A. Alzati and J. Sierra get a bound of quadrics for $\mathbf{N}_{2,2}$ as paying attention to the structures of the rational map associated to the linear system of quadrics defining X , which coincides with our bound LB_2 ([2]). They also derive a degree bound in terms of codimension e , $\binom{d}{2} \leq \binom{2e-1}{e-1}$ whose asymptotic behavior is $2^e/\sqrt[4]{\pi e}$ and describe the equality condition: *this holds if and only if the equality of LB_2 holds*. From this theorem, we have a rigid condition on degree of X in case of $p = 2$ as if we get some rigidity when $p = 1, e - 1$, and e (see Remark 3.8 and Theorem 3.14). Using our inner projection method (e.g. Corollary 3.7), we could improve this degree bound a little as follows:

$$\binom{d+2-p}{2} \leq \binom{2e+3-2p}{e+1-p} \quad , \quad d \sim 2^{e+2-p}/\sqrt[4]{\pi e} \quad (\text{as } e \text{ getting sufficiently large})$$

under the assumption of property $\mathbf{N}_{2,p}$ ($p \geq 2$) of X .

Example 3.10. It would be interesting to know that if $e \leq \beta_{0,2}(I_X) < 2e - 1$ then X has always at least a syzygy of defining equations which is not linear because property $\mathbf{N}_{2,2}$ does not hold for X . For example, let C be the general embedding of degree 19 in \mathbb{P}^7 of genus 12. Then C is a smooth arithmetically Cohen-Macaulay curve which is cut out scheme-theoretically by 9 quadrics, but the homogeneous ideal I_C is generated by 9 quadrics and 2 cubics (see [21] for details). These quadric generators should have at least a syzygy of high degree as well as linear syzygies.

Example 3.11. (Veronese embedding $v_d(\mathbb{P}^n)$) It is well-known that $v_d(\mathbb{P}^n)$ satisfies property $\mathbf{N}_{2,3d-3}$ and fails property $\mathbf{N}_{2,3d-2}$ for $n \geq 2$ and $d \geq 3$ ([10], [23]). We can also verify the failure of property $\mathbf{N}_{2,p}$ of Veronese embedding $X = v_d(\mathbb{P}^n)$ for some cases by using this low bound LB_p . For example, when $n = 2, d = 3, p = 3d - 2 = 7$, we get $\beta_{0,2}(I_X) = 27$ and $\text{LB}_7 = 7 \cdot 7 - \binom{7}{2} = 28$. Therefore, $v_3(\mathbb{P}^2)$ fails to satisfy $\mathbf{N}_{2,7}$. Similarly, $v_2(\mathbb{P}^3)$ fails property $\mathbf{N}_{2,6}$. However, it does not give the reason why $v_3(\mathbb{P}^3)$ does fail to be $\mathbf{N}_{2,7}$ for the case $n = d = 3, p = 3d - 2 = 7, e = 16$, because $\beta_{0,2}(I_X) = 126 > 91 = \text{LB}_7$. For such p in the middle area of $1 \leq p \leq e$, LB_p seems not to give quite sufficient information for property $\mathbf{N}_{2,p}$, while it may be more effective to decide $\mathbf{N}_{2,p}$ of a given variety for rather large p among $1 \leq p \leq e$.

Now, we proceed to investigate the depth of inner projections to understand the shape of the Betti diagram and Castelnuovo-Mumford regularity. This result looks very surprising when we compare this with the outer projection case.

Theorem 3.12. (The depth of inner projections) *Let $X \subset \mathbb{P}^N$ be a non-degenerate reduced and connected subscheme cut out by quadrics (i.e. $\mathbf{N}_{2,1}$). Consider the inner projection $\pi_q : X \dashrightarrow X_q \subset \mathbb{P}^{N-1}$ from a smooth point $q \in X$. Then,*

- (a) *the projective dimension of S/I_{X_q} , $\text{pd}_S(S/I_{X_q}) = \text{pd}_R(R/I_X) - 1$;*
- (b) *$\text{depth}_R(X) = \text{depth}_S(X_q)$. In particular, X is arithmetically Cohen-Macaulay if and only if so is X_q .*

Proof. (a) We know $\text{pd}_R(R/I_X) \geq e = \text{codim} X$. Let l be $\text{pd}_R(R/I_X)$ (so, $l \geq e$), and $j_0 = \max\{j \mid \text{Tor}_l^R(R/I_X)_{l+j} \neq 0\}$.

Case 1) Non-Cohen Macaulay case (i.e. $l = e + \alpha$, $\alpha \geq 1$) :

First of all, we have the following diagram from the exact sequence (3.1):

$$\begin{array}{ccccccc}
 i \text{ of } \operatorname{Tor}_i^S(S/I_{X_q}) & 0 \rightarrow & I_{X_q} & \rightarrow & I_X & \rightarrow & I_X/I_{X_q} \rightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 1 & & \square & & \square & & S(-2)^e \oplus S(-3)^e \oplus \cdots \\
 2 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 e & & \square & & \square & & S(-e-1) \oplus S(-e-2) \oplus \cdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 e+1 & & \square & \cong & \square & & 0 \\
 \vdots & & \vdots & & \vdots & & \vdots \\
 l = e + \alpha & & \blacksquare & \cong & \blacksquare & & 0 \\
 \blacksquare : \text{vanished} & & & & & &
 \end{array}$$

From this diagram, we get $\operatorname{Tor}_l^S(R/I_X) \cong \operatorname{Tor}_l^S(S/I_{X_q})$ as finite k -vector spaces. Since $\operatorname{Tor}_{l+1}^R(R/I_X)_{l+1+j} = 0$ for all j ($\because \operatorname{pd}_R(R/I_X) = l$) and $\operatorname{Tor}_l^R(R/I_X)_{l+j} = 0$ for $j > j_0$, we can observe using the mapping cone sequence (2.1) that

$$\cdots \xrightarrow{\times x_0} \operatorname{Tor}_l^S(R/I_X)_{l+j} \xrightarrow{\times x_0} \cdots \xrightarrow{\times x_0} \operatorname{Tor}_l^S(R/I_X)_{l+j_0} \xrightarrow{\sim} \cdots$$

So we have $\operatorname{Tor}_l^S(R/I_X) = 0$, because it is finite dimensional. This means $\operatorname{Tor}_l^S(S/I_{X_q}) \cong \operatorname{Tor}_l^S(R/I_X) = 0$.

Next, we claim that $\operatorname{Tor}_{l-1}^S(S/I_{X_q}) \neq 0$, which implies $\operatorname{pd}_S(S/I_{X_q}) = l - 1$. If $\alpha \geq 2$, then we have $\operatorname{Tor}_{l-1}^S(S/I_{X_q}) \simeq \operatorname{Tor}_{l-1}^S(R/I_X)$. Since $\operatorname{Tor}_l^S(R/I_X) = 0$, we have a nontrivial kernel of the $\times x_0$ map in I_X from the mapping cone sequence (2.1)

$$\begin{array}{ccccc}
 0 \rightarrow & \operatorname{Tor}_l^R(R/I_X)_{l+j_0} \hookrightarrow & \operatorname{Tor}_{l-1}^S(R/I_X)_{l-1+j_0} & \xrightarrow{\times x_0} & \operatorname{Tor}_{l-1}^S(R/I_X)_{l-1+j_0+1} \cdots (*) \\
 & \Downarrow & \parallel \wr & & \parallel \wr \\
 & 0 & \operatorname{Tor}_{l-2}^S(I_X)_{l-1+j_0} & & \operatorname{Tor}_{l-2}^S(I_X)_{l-1+j_0+1}
 \end{array}$$

This implies $\operatorname{Tor}_{l-1}^S(R/I_X) \simeq \operatorname{Tor}_{l-1}^S(S/I_{X_q}) \neq 0$ as wished. So, let us focus on the case $\alpha = 1$ and so, $l = e + 1$. Consider the following sequence and commutative diagram:

$$\operatorname{Tor}_{l-1}^S(I_X/I_{X_q}) = 0 \rightarrow \operatorname{Tor}_{l-2}^S(I_{X_q}) \rightarrow \operatorname{Tor}_{l-2}^S(I_X) \rightarrow \operatorname{Tor}_{l-2}^S(I_X/I_{X_q}) \rightarrow \cdots$$

$$\begin{array}{ccccc}
 \operatorname{Tor}_{l-2}^S(I_X)_{e+j_0} \simeq & S(-e-j_0)^\square \otimes k & \xrightarrow{f_{e+j_0}} & S(-e-j_0) \otimes k & \simeq \operatorname{Tor}_{l-2}^S(I_X/I_{X_q})_{e+j_0} \\
 & h \downarrow \text{not injective} & & g \downarrow \wr & \\
 & S(-c) \otimes k & \xrightarrow{\sim} & S(-c) \otimes k &
 \end{array}$$

where h, g are induced by the multiplication of x_0^n and g is an isomorphism. To check $\operatorname{Tor}_{l-1}^S(S/I_{X_q}) \cong \operatorname{Tor}_{l-2}^S(I_{X_q}) \neq 0$, it is enough to show that $f : \operatorname{Tor}_{l-2}^S(I_X) \rightarrow \operatorname{Tor}_{l-2}^S(I_X/I_{X_q})$ is not injective because $\operatorname{Tor}_{l-1}^S(I_X/I_{X_q}) = 0$. Now let me explain why f be not injective. We get the below isomorphism map for $c \gg 0$ because $\operatorname{Tor}_i^S(I_{X_q})$ are finite-dimensional graded vector spaces and also h is not injective by (*). From Remark (2.7), this diagram commutes and f_{e+j_0} has a nontrivial kernel. Hence $f : \operatorname{Tor}_{l-2}^S(I_X) \rightarrow \operatorname{Tor}_{l-2}^S(I_X/I_{X_q})$ is not injective and $\operatorname{Tor}_{l-1}^S(S/I_{X_q})_{e+j_0} = \operatorname{Tor}_{l-2}^S(I_{X_q})_{e+j_0} \neq 0$.

Case 2) Cohen-Macaulay case (i.e. $l = e$, $\alpha = 0$) : In this case, we have the long exact sequence on Tor as follows:

$$\begin{array}{ccccccc} & & & & & S(-e-1) \oplus & \\ & & & & & \downarrow f & \\ 0 = \mathrm{Tor}_e^S(I_X/I_{X_q}) & \rightarrow & \mathrm{Tor}_{e-1}^S(I_{X_q}) & \rightarrow & \mathrm{Tor}_{e-1}^S(I_X) & \rightarrow & S(-e-2) \oplus \\ & & & & & \downarrow & \\ & & & & & \dots & \end{array}$$

Since $\mathrm{pd}_R(R/I_X) = e$, $\mathrm{Tor}_{e+1}^R(R/I_X) = 0$ and we have an injection

$$\mathrm{Tor}_{e-1}^S(I_X)_{e+j} = \mathrm{Tor}_e^S(R/I_X)_{e+j} \xrightarrow{\times x_q^n} \mathrm{Tor}_e^S(R/I_X)_{e+j+n} = \mathrm{Tor}_{e-1}^S(I_X)_{e+j+n}$$

for any j, n from the mapping cone sequence (2.1). By almost same argument using the commuting diagram as **Case 1)**, $\alpha = 1$, we can conclude that $f : \mathrm{Tor}_{e-1}^S(I_X) \rightarrow \mathrm{Tor}_{e-1}^S(I_X/I_{X_q})$ is injective and $\mathrm{Tor}_e^S(S/I_{X_q}) = 0$. So, this means $\mathrm{pd}_S(S/I_{X_q}) \leq e-1$. But we know $\mathrm{pd}_S(S/I_{X_q}) \geq \mathrm{codim}(X_q) = e-1$, therefore $\mathrm{pd}_S(S/I_{X_q}) = e-1$. On the other hand, (a) implies that $\mathrm{depth}_R(X) = \mathrm{depth}_S(X_q)$ by Auslander-Buchsbaum formula. \square

Remark 3.13. Let $X \subset \mathbb{P}^n$ be a reduced scheme satisfying property $\mathbf{N}_{2,p}$ ($p \geq 2$). Let $\Sigma_q(X) = \{x \in X \mid \pi_q^{-1}(\pi_q(x)) \text{ has length } \geq 2\}$ is the *secant locus* of the outer projection. We would like to remark that $\mathrm{depth}(X_q) = \min\{\mathrm{depth}(X), \dim \Sigma_q(X) + 2\}$ for a smooth scheme X ([1], [24]). On the other hand, it would be interesting to ask the following question: Is there an example such that $\mathrm{depth}(X_q) \neq \mathrm{depth}(X)$ for inner projections?

As an interesting application of above results, we can also prove the following rigidity theorem for the *extremal* (i.e. $p = e$) and *next to extremal* (i.e. $p = e-1$) cases of property $\mathbf{N}_{2,p}$ of the varieties by using Corollary 3.3, Proposition 3.5 and Theorem 3.12.

Theorem 3.14. (Rigidity for property $\mathbf{N}_{2,p}$) *Let X be a non-degenerate r -dimensional variety (i.e. irreducible, reduced) in \mathbb{P}^{r+e} , $e = \mathrm{codim}(X, \mathbb{P}^{r+e})$.*

- (a) *(extremal case) X satisfies property $\mathbf{N}_{2,e}$ if and only if X is a minimal degree variety, i.e. a whole linear space \mathbb{P}^{r+e} , a quadric hypersurface, a cone of the Veronese surface in \mathbb{P}^5 , or rational normal scrolls;*
- (b) *(next to extremal case) X fails property $\mathbf{N}_{2,e}$ but satisfies $\mathbf{N}_{2,e-1}$ if and only if X is a del Pezzo variety, i.e. X is arithmetically Cohen-Macaulay (abbr. ACM) and is of next to minimal degree.*

Proof. Let $\delta(X) := \deg(X) - \mathrm{codim}(X)$ for any subvariety $X \subset \mathbb{P}^{r+e}$. Note that $\delta(X_q) = \delta(X)$ under an inner projection from a smooth point $q \in X$. Take successive inner projections from smooth points. Call the images (Zariski closure) $X = X^{(0)}, X^{(1)}, \dots, X^{(e-2)}$ and $I^{(i)}$ for the elimination ideal of $I^{(i-1)}$ cutting out $X^{(i)}$ scheme-theoretically. By Corollary 3.3 we know that this $X^{(e-2)}$ has codim 2 and have property $\mathbf{N}_{2,2}$ for (a) (and $\mathbf{N}_{2,1}$ for (b), respectively). Because $X^{(e-1)}$ is a hypersurface, by Proposition 3.5 the possible $\beta_{0,2}(I^{(e-2)}) = 2$ or 3. For the case (a), take an inner projection once more and then $X^{(e-1)}$ still satisfies property $\mathbf{N}_{2,1}$, i.e. an (irreducible) quadric hypersurface. So,

$$\beta_{0,2}(I^{(e-1)}) = 1, \beta_{0,2}(I^{(e-2)}) = 1 + 2 = 3 \text{ and } \delta(X) = \delta(X^{(e-1)}) = 1.$$

That is, X is of minimal degree. In the case of (b), $X^{(e-2)}$ is a complete intersection of two quadrics in \mathbb{P}^{r+2} and $X^{(e-1)}$ should be a cubic hypersurface.

In particular, the projective dimension of $X^{(e-2)}$ is equal to $2 = \text{pd}_R(R/I_X) - (e-2)$ by Theorem 3.12. Therefore,

$$\beta_{0,2}(I^{(e-1)}) = 0, \beta_{0,2}(I^{(e-2)}) = 2, \delta(X) = \delta(X^{(e-1)}) = 2 \text{ and } \text{pd}_R(R/I_X) = e,$$

which means X is arithmetically Cohen-Macaulay and of next to minimal degree with $H^0(\mathcal{I}_X(2)) = \frac{(e+2)(e-1)}{2}$. By the well-known classification of varieties of next to minimal degree, X is a del Pezzo variety. On the other hand, the curve section C of a del Pezzo variety X is either an elliptic normal curve or a projection of a rational normal curve from a point in $\text{Sec}(C) \setminus C$. Since X and C have the same Betti diagram, X satisfies property $\mathbf{N}_{2,e-1}$ but fail property $\mathbf{N}_{2,e}$. \square

Remark 3.15. There are some remarks on Theorem 3.14.

- (a) For the smooth projectively normal variety X , M. Green's $\mathcal{K}_{p,1}$ -theorem ([15]) gives a necessary condition on Theorem 3.14 (b) (i.e. X is either a variety of next to minimal degree or a divisor on a minimal degree). Using our Corollary 3.3 and *Depth* theorem 3.12, we could obtain the results on both $\deg(X) = \text{codim} X + 2$ and ACM property and show the rigidity on next to extremal case for *any* (not necessarily projectively normal) variety X .
- (b) Classically, normal del Pezzo varieties were classified by T. Fujita [13]. And every non-normal del Pezzo variety X ([6],[13]) comes from outer projection of a minimal degree variety \tilde{X} from a point q in $\text{Sec}(\tilde{X}) \setminus \tilde{X}$ satisfying $\dim \Sigma_q(\tilde{X}) = \dim \tilde{X} - 1$ (see remark 3.13). Note that the dimension of the secant locus $\Sigma_q(\tilde{X})$ varies as q moves in $\text{Sec}(\tilde{X}) \setminus \tilde{X}$. Thus one can try to classify the non-normal del Pezzo varieties by the type of the secant loci $\Sigma_q(\tilde{X})$. This is recently classified in [5] such that there are only 8 types of non-normal del Pezzo varieties which are not cones. For example, we find projections of a smooth cubic surface scroll $S(1,2)$ in \mathbb{P}^4 from any $q \in \mathbb{P}^4 \setminus S(1,2)$ or projections of a smooth 3-fold scroll $S(1,1,c)$ in \mathbb{P}^{c+4} with $c > 1$ from any $q \in \langle S(1,1) \rangle \setminus S(1,1,c)$, etc.

Furthermore, let's consider the followig category. (We say that an algebraic set $X = \cup X_i$ is *connected in codimension 1* if all the component X_i 's can be arranged in such a way that every $X_i \cap X_{i+1}$ is of codimension 1 in X .)

{equidimensional, connected in codimension 1, reduced subschemes in \mathbb{P}^{r+e} } $\dots (*)$

In the category $(*)$, we have natural notions of $\dim X$ and $\deg(X)$ which is the sum of degrees of X_i 's. And as in the category of *varieties*, we also have the 'basic' inequality of degree, i.e. $\deg(X) \geq \text{codim} X + 1$, so it is worthwhile to think of 'minimal degree' or 'next to minimal degree' in this category.

Using same methods, the Theorem 3.14 can be easily extended for this category. We call X (or the sequence) *linearly joined* whenever all the components can be ordered X_1, X_2, \dots, X_k so that for each i , $(X_1 \cup \dots \cup X_i) \cap X_{i+1} = \text{span}(X_1 \cup \dots \cup X_i) \cap \text{span}(X_{i+1})$. Then, we have a corollary as follows:

Corollary 3.16. *Let X be a non-degenerate subscheme in the category $(*)$ with $e = \text{codim}(X, \mathbb{P}^{r+e})$.*

- (a) (extremal case) X satisfies property $\mathbf{N}_{2,e}$ if and only if X is 2-regular, i.e. the linearly joined sequences of r -dimensional minimal degree varieties;
- (b) (next to extremal case) If X fails property $\mathbf{N}_{2,e}$ but satisfies $\mathbf{N}_{2,e-1}$, then X is arithmetically Cohen-Macaulay and is of next to minimal degree.

Proof. For (a) we can also project X to a hyperquadric $X^{(e-1)}$ similarly (In this case $X^{(e-1)}$ is reducible, i.e. a union of two r -linear planes and every component of X degenerates into this linear subspaces). So $X^{(e-1)}$ is ACM, and from our Depth theorem 3.12 X is also ACM, eventually 2-regular. We also have a similar result for the case of (b) by same arguments; X becomes ACM and of next to minimal degree subscheme with $h^0(\mathcal{I}_X(2)) = \frac{(e+2)(e-1)}{2}$ in this category. \square

Remark 3.17. For the ‘rigidity’ of property $\mathbf{N}_{2,p}$ for $p = e$, D. Eisenbud et al. proved the same theorem for the category of algebraic set more generally in [10]. The (geometric) classification of 2-regularity is well-known for varieties, and for general algebraic sets it is given in [11]. We reprove this rigidity (case (a)) using our inner projection method and for next to extremal case (b) we also get similar classification for the subschemes in the category (*).

4. SOME EXAMPLES AND OPEN QUESTIONS

It seems to be quite natural to find out a good inner projection as we move the point $q \in X$ in many aspects. What happens to inner projections from singular points? During the discussions with P. Schenzel, we have the following example;

Example 4.1. (Projection from a *singular* point, discussion with P. Schenzel)
Let us consider a singular surface in \mathbb{P}^{14} by Segre embedding of quadric in \mathbb{P}^2 and singular quintic rational curve in \mathbb{P}^4 .

	0	1	2	\dots	i
2	70	475	1605	\dots	$\beta_{i,2}$
3	-	-	11	\dots	$\beta_{i,3}$

\Rightarrow property $\mathbf{N}_{2,2}$ holds (Note: ‘-’ means “zero” in Betti diagram).

1) inner projection from a smooth point

	0	1	2	\dots	i
2	58	351	1035	\dots	$\beta_{i,2}$
3	-	1	19	\dots	$\beta_{i,3}$

\Rightarrow property $\mathbf{N}_{2,1}$ holds as the main Theorem says.

2) inner projection from any point in the singular locus

	0	1	2	\dots	i
2	59	362	1089	\dots	$\beta_{i,2}$
3	-	-	10	\dots	$\beta_{i,3}$

\Rightarrow still property $\mathbf{N}_{2,2}$ holds!

Example 4.2. Let X be the Grassmannian $\mathbb{G}(2, 4)$ in \mathbb{P}^9 , 6- dimensional del Pezzo variety of degree 5 whose Betti diagram is

	0	1	2	\dots	i
2	5	5	-	-	$\beta_{i,2}$
3	-	-	1	-	$\beta_{i,3}$

and property $\mathbf{N}_{2,2}$ is satisfied. Since it is homogeneous and covered by lines, so we can choose any (smooth) point q in X and a line ℓ through q in X . Then the projection X_q is a complete intersection of two quadrics in \mathbb{P}^8 (property $\mathbf{N}_{2,1}$) and $q' = \pi_q(\ell)$ becomes a singularity of multiplicity 2 in X_q . If we project this one more from q' , then the projected image becomes a quadric hypersurface in \mathbb{P}^7 still satisfying property $\mathbf{N}_{2,1}$.

Question 4.3. (Inner projection from a singular point) Assume that X be a non-degenerate projective scheme with $\mathbf{N}_{2,p}$. If $q \in X$ is singular, we could expect that the inner projection from q has more complicate aspects, but shows better behavior still satisfying $\mathbf{N}_{2,p}$ in many experimental data. What can happen to the projection from singular locus in general?

Next, we consider Problem (a) of the introduction in general. Let X be a non-degenerate subscheme with property $\mathbf{N}_{2,p}$. If ℓ meets X but is not contained in X , then we can regard the projection π_ℓ as the composition of two simple projections from points q_1, q_2 . Furthermore, if such ℓ meets X at smooth two points, then $X_\ell = \pi_\ell(X \setminus \ell)$ satisfies property $\mathbf{N}_{2,p-2}$ by our main Theorem.

But not the case of $\ell \subset X$ we can treat simply, because $q_2 = \pi_{q_1}(\ell)$ might be a singular point even if $\pi_\ell = \pi_{q_2} \circ \pi_{q_1}$. In this case, we give an interesting example showing that the Betti numbers of $X_\ell = \pi_\ell(X \setminus \ell)$ are related to the geometry of the line inside X .

Example 4.4. (Projection from a line *inside* the variety) Consider the Segre embedding $X = \sigma(\mathbb{P}^2 \times \mathbb{P}^4)$, 6-fold of degree 15 in \mathbb{P}^{14} having property $\mathbf{N}_{2,3}$ whose Betti diagram is

	0	1	2	3	4	5	6	7	\dots	i
2	30	120	210	168	50	-	-	-	-	$\beta_{i,2}$
3	-	-	-	50	120	105	40	6	-	$\beta_{i,3}$

If we take ℓ_1 as the line $\sigma(\{pt\} \times \ell)$ in X , then the image X_{ℓ_1} is the intersection of two cones $\langle \sigma(\mathbb{P}^1 \times \mathbb{P}^4), \mathbb{P}^2 \rangle$ and $\langle \mathbb{P}^3, \sigma(\mathbb{P}^2 \times \mathbb{P}^2) \rangle$ which is a 6-fold of degree 12 in \mathbb{P}^{12} satisfying property $\mathbf{N}_{2,2}$ with the Betti diagram

	0	1	2	3	4	5	\dots	i
2	16	40	30	4	-	-	-	$\beta_{i,2}$
3	-	-	20	40	24	5	-	$\beta_{i,3}$

On the other hand, in case of the line $\ell_2 = \sigma(\ell \times \{pt\})$, X_{ℓ_2} is a 6-dimensional cone $\langle \{pt\}, \sigma(\mathbb{P}^2 \times \mathbb{P}^3) \rangle$ of degree 10 in \mathbb{P}^{12} and has its own Betti diagram

	0	1	2	3	4	5	...	i
2	18	52	60	24	-	-	-	$\beta_{i,2}$
3	-	-	-	10	12	3	-	$\beta_{i,3}$

with property $\mathbf{N}_{2,3}$. Note that the dimension of the span $\langle \cup_{q \in \ell_1} T_q X \rangle$ of tangent spaces along ℓ_1 is 8, but $\dim \langle \cup_{q \in \ell_2} T_q X \rangle = 10$ (i.e. the tangent spaces change more variously along ℓ_2 than ℓ_1). So, it is expected that ℓ_2 is geometrically less movable than ℓ_1 inside X and X_{ℓ_2} has more linear syzygies.

Question 4.5. (Inner projection from a subvariety) Let X be a nondegenerate reduced, connected scheme in \mathbb{P}^N satisfying property $\mathbf{N}_{2,p}$, $p > 1$ which is not necessarily linearly normal. Consider the inner projection from a line $\ell \subset X$. Is it true that $\overline{\pi_\ell(X \setminus \ell)}$ satisfies $\mathbf{N}_{2,p-2}$? How does the *local* geometry of ℓ in X effect to the syzygies of $\overline{\pi_\ell(X \setminus \ell)}$? More generally, how about the projection from a subvariety Y of X ? The projection from Y is defined by the projection from $\Lambda := \langle Y \rangle$, the linear span of Y ([4]). Say $\dim \Lambda = t < p$. Does X_Λ in \mathbb{P}^{N-t-1} satisfy property $\mathbf{N}_{2,p-t-1}$ in general as raised in the problem list (a) in the introduction?

For the sake of Question 4.5, we expect to need developing the graded mapping cone theorem and partial elimination module theory for multivariate case and calculating on the syzygies of those partial elimination modules by Gröbner basis theory for graded modules. See [18] for basic settings and some partial results for bivariate case.

Finally, we have the following question as raised in Remark 3.17.

Question 4.6. (Classification of some 3-regular, ACM schemes) We showed in Section 3 that if a r -equidimensional, reduced and connected in codimension 1 subscheme X in \mathbb{P}^{r+e} fails property $\mathbf{N}_{2,e}$ but satisfies $\mathbf{N}_{2,e-1}$, then it is a ACM, 3-regular scheme of next to minimal degree (i.e. $\deg(X) = \text{codim} X + 2$) with $h^0(J_X(2)) = \frac{(e+2)(e-1)}{2}$. Further, L.T. Hoa ([20]) gave the complete graded Betti numbers as follows:

$$0 \rightarrow R(-e-2) \rightarrow R^{\beta_{e-2,2}}(-e) \rightarrow R^{\beta_{e-3,2}}(-e+1) \rightarrow \cdots \rightarrow R^{\beta_{0,2}}(-2) \rightarrow I_X \rightarrow 0 \cdots (**)$$

where $\beta_{i,2} = (i+1)\binom{e+1}{i+2} - \binom{e}{i}$ for $0 \leq i \leq e-2$. Thus, as like the classification of reduced 2-regular projective schemes (see [11]), among all the equidimensional, reduced and connected in codimension 1 subschemes, it would be very interesting to classify or give some geometric descriptions for all 3-regular, ACM, and next to minimal degree projective schemes whose Betti diagram is given by (**).

REFERENCES

- [1] J. Ahn and S. Kwak, *Graded mapping cone theorem, multiseccants and syzygies*, submitted (2008)
- [2] A. Alzati and J. Sierra, *A bound on the degree of schemes defined by quadratic equations*, preprint.
- [3] I. Bauer, *Inner projections of algebraic surfaces: a finiteness result*, J. Reine Angew. Math. 460, 1–13 (1995).
- [4] M. Beltrametti, A. Howard, M. Schneider and A. Sommese, *Projections from subvarieties*, Complex Analysis and Algebraic Geometry (T. Peternell, F.-O. Schreyer, eds.), A Volume in Memory of Michael Schneider, (2000), 71–107.

- [5] M. Brodmann and E. Park, *On varieties of almost minimal degree I: Secant loci of rational normal scrolls*, arXiv:0808.0090v1[math.AG]
- [6] M. Brodmann and P. Schenzel, *Arithmetic properties of projective varieties of almost minimal degree*, J. Algebraic Geom. 16 (2007), 347–400.
- [7] A. Calabri and C. Ciliberto, *On special projections of varieties: epitome to a theorem of Beniamino Segre*, Adv. Geom. 1 (2001), no. 1, 97–106.
- [8] Y. Choi, S. Kwak and P-L Kang, *Higher linear syzygies of inner projections*, J. Algebra 305 (2006), 859–876.
- [9] Y. Choi, S. Kwak and E. Park, *On syzygies of non-complete embedding of projective varieties*, Math. Zeitschrift 258, no. 2 (2008), 463–475.
- [10] D. Eisenbud, M. Green, K. Hulek and S. Popescu, *Restriction linear syzygies: algebra and geometry*, Compositio Math. 141 (2005), 1460–1478.
- [11] D. Eisenbud, M. Green, K. Hulek and S. Popescu, *Small schemes and varieties of minimal degree*, Amer. J. Math. 128 (2006), no. 6, 1363–1389.
- [12] D. Eisenbud, C. Huneke, B. Ulrich, *The regularity of Tor and graded Betti numbers*, Amer. J. Math. 128 (2006), no. 3, 573–605.
- [13] T. Fujita, *Classification theories of polarized varieties*, Cambridge University Press, Cambridge, (1990).
- [14] H. Flenner, L. O’Carroll, W. Vogel, *Joins and intersections*, Springer-Verlag, Berlin, (1999).
- [15] M. Green, *Koszul cohomology and the geometry of projective varieties*, J. Differential Geom. 19 (1984), 125–171.
- [16] M. Green, *Generic Initial Ideals*, in Six lectures on Commutative Algebra, (Elias J., Giral J.M., Miró-Roig, R.M., Zarzuela S., eds.), Progress in Mathematics **166**, Birkhäuser, 1998, 119–186.
- [17] M. Green and R. Lazarsfeld, *Some results on the syzygies of finite sets and algebraic curves*, Compositio Math. 67 (1988), no. 3, 301–314.
- [18] K. Han and S. Kwak, *Projections from lines : algebraic and geometric properties*, in preparation.
- [19] R. Hartshorne, *Algebraic geometry*, Graduate text, Springer Verlag,
- [20] L.T. Hoa, *On minimal free resolutions of projective varieties of degree = codimension + 2*, J. Pure Appl. Algebra, 87 (1993), 241–250.
- [21] S. Katz, *Arithmetically Cohen-Macaulay curves cut out by quadrics*, Computational algebraic geometry and commutative algebra (Cortona, 1991), 257–263, Sympos. Math., XXXIV, Cambridge Univ. Press, Cambridge, 1993.
- [22] S. Kwak and E. Park *Some effects of property N_p on the higher normality and defining equations of nonlinearly normal varieties*, J. Reine Angew. Math. 582 (2005), 87–105.
- [23] G. Ottaviani and R. Paoletti, *Syzygies of Veronese embeddings*, Compositio Math., 125 (2001), 31–37.
- [24] E. Park, *On secant loci and simple linear projections of some projective varieties*, preprint
- [25] M. Reid, *Graded rings and birational geometry*, in Proc. of algebraic geometry symposium (Kinosaki, Oct 2000), K. Ohno (Ed.), 1–72.
- [26] B. Segre, *On the locus of points from which an algebraic variety is projected multiply*, Proc. Phys.-Math. Soc. Japan Ser. III, 18 (1936), 425–426.
- [27] A. Sommese, *Hyperplane sections of projective surfaces I, The adjunction mapping*, Duke Math. J. 46 (1979), no. 2, 377–401.
- [28] F. L. Zak, *Projective invariants of quadratic embedding*, Math. Ann. 313 (1999), 507–545.

ALGEBRAIC STRUCTURE AND ITS APPLICATIONS RESEARCH CENTER(ASARC), DEPARTMENT OF MATHEMATICS, KOREA ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY, 373-1 GUSUNG-DONG, YUSUNG-GU, DAEJEON, KOREA

E-mail address: han.kangjin@kaist.ac.kr

DEPARTMENT OF MATHEMATICS, KOREA ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY, 373-1 GUSUNG-DONG, YUSUNG-GU, DAEJEON, KOREA

E-mail address: skwak@kaist.ac.kr